On Meissner Effect and Superfluid Density in Superconductors

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As the most successful microscopic superconductivity theory, Bardeen-Cooper-Schrieffer(BCS) theory has a very peculiar prediction: at zero temperature, only a fraction of electrons within an energy shell form Cooper pair and condense, but all electrons participate to form a macroscopic superfluid and contribute to the superfluid density (inverse square of penetration depth). Very recently, this prediction was challenged by directly measuring the penetration depth upon doping in overdoped cuprates. [1] Here, we show that such a counter-intuitive prediction of BCS theory is not right. The key point is to disentangle two fundamental concepts in superconductors: plasma frequency and superfluid density, which were thought to be equal for more than half a century. In our theory, superfluid density is determined only by paired electrons while plasma frequency by all electrons. As a consequence, the widely used technique to obtain superfluid density through optical conductivity, based on Ferrell-Glover-Tinkham sum rule, measures only plasma frequency but not superfluid density. Our theory has been evidenced by existed anomalous scaling laws in different experiments.

Superconductivity has two fascinating properties: perfect conductivity and perfect diamagnetism (Meissner effect). These two properties were first phenomenologically described by two London equations. [2] But the microscopic theory was missing for many years until Bardeen-Cooper-Schrieffer(BCS)'s milestone work. [3] BCS theory has been verified by numerous experiments and finally accepted as a successful superconductivity theory. Although there has been some objective voice about its ability to explain Meissner effect, [4] most people believe BCS is right. According to BCS theory, only fermions within energy shell $(-\omega_D, \omega_D)$ (ω_D is Debye frequency) pair, but the superfluid density ρ_s (inverse square of penetration depth) is contributed by all electrons at zero temperature, i.e. $\rho_s^{\mathrm{BCS}} = \frac{ne^2}{m}$. [3, 5] This counter-intuitive result is argued to be caused by the energy gap and widely accepted. Recently, by optical experiments, people have found a famous scaling law (Homes' law [6]) $\rho_s \propto \sigma_n T_c$ (σ_n the normal state dc conductivity slightly above T_c) which seems to violate the BCS's prediction. But this violation was attributed to the consequence of dirty superconductors in literatures. [7] Another violation is the Uemura's law, [8] $\rho_s \propto T_c$ in underdoped cuprates, which however was thought to be a result of carrier density proportional to T_c. [9] Very recently, Bozovic et al. has found another violation: $\rho_s \propto T_c(T_c^2)$ for high(low)-T_c samples in overdoped cuprates, [1, 10] which was though to be described well by BCS theory. This striking observation, however, cannot be simply described by dirty superconductors (see later discussions and also Ref. [1]). More severely, Bozovic's observation is in contrast with Homes' law as long as BCS's prediction is right. So apparently, if both experiments are right, BCS's prediction must be wrong. As a result, how to rescue the BCS theory or just throwing it away is a fundamental question. In this work, we show that BCS theory is essentially right, but needs a small correction which calls for a new understanding of superfluid density different from BCS's original way.

The celebrated proof of Meissner effect in BCS theory was based on the linear response calculation of current-current correlation $K_{ij}(\omega = 0, \mathbf{q}) = \left(\frac{q_i q_j}{\mathbf{q}^2} - \delta_{ij}\right)\Pi(\mathbf{q}^2)$, where nonzero $\Pi(\mathbf{q}^2 \to 0)$ gives the superfluid density and thus indicates the Meissner effect. [3, 5] Historically, this strategy was first proposed by Schafroth [11, 12] and later widely used. Shortly after BCS's work, people paid much effort to solve the gauge invariance problem [13–15] but no one doubted the applicability of Schafroth's proposal. Until today, people have been believing it to be correct for more than sixty years. In fact, from $J_i(\mathbf{q}) = \Pi_{ij}(\mathbf{q})A_j(\mathbf{q})$, we can only obtain

$$\mathbf{q} \times \mathbf{J} = -\Pi(\mathbf{q}^2)\mathbf{q} \times \mathbf{A}, \tag{1}$$

which seems to be equivalent to the second London equation $\nabla \times \mathbf{J} = -\Pi(\mathbf{q}^2)\mathbf{B}$. Here, however, we point out this equivalence is problematic in $\mathbf{q} = 0$ limit since Eq. 1 only leads to $0 = 0$ and thus *nonzero* $\Pi(\mathbf{q}^2 \to 0)$ *does not explain Meissner effect at all*. This subtlety is caused by the fact: nonzero $\mathbf{B}$ with $\mathbf{q} = 0$ cannot be obtained naively by Fourier transformation of $\mathbf{B} = \nabla \times \mathbf{A}$, i.e. $\mathbf{B}(\mathbf{q}) = i\mathbf{q} \times \mathbf{A}(\mathbf{q})$.

On the other hand, $K_{ij}(\omega \to 0, \mathbf{q} = 0)$ was also widely applied to indicate Meissner effect, known as the Farrell-Glover-Tinkham(FGT) sum rule[16–18]: the missing spectral in real part of longitudinal optical conductivity $\sigma_i(\omega) = \frac{1}{-i\omega}K_{ii}(\omega, \mathbf{q} = 0)$ transfers to zero frequency as a delta peak $A\delta(\omega)$, which is recognized as the superfluid density. However, this theory is based on two assumptions: $K_{ii}(\omega \to 0, \mathbf{q} = 0) = -\Pi(\mathbf{q}^2 \to 0)$ and $\Pi(\mathbf{q}^2 \to 0)$ equals superfluid density. Since the second assumption has been argued to be incorrect, we are led to another conclusion: *the missing spectral in* $\sigma(\omega)$, *or condensed spectral* $A\delta(\omega)$ *does not explain Meissner effect at all*.

In order to support the above statements, we provide a counterexample: perfect conductor. Consider the clas-

sical Drude model without scattering rate (infinite lifetime), the optical conductivity is $\sigma(\omega) = \frac{ne^2}{-i\omega m}$, corresponding to a delta peak in its real part: $\mathrm{Re}[\sigma(\omega)] = \frac{ne^2}{m}\delta(\omega)$. This is not others, but the condensed spectral. Does it mean Meissner effect? The answer is clearly no. It just gives us the plasma frequency $\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}$, which can be understood as the photon mass $\omega^2 = k^2 + \omega_p^2$. As a result, light with frequency $\omega < \omega_p$ cannot propagate in the bulk. But it does not exclude the possibility of static magnetic field penetration. Anderson had already noticed such a mass mechanism [19] but it was not taken seriously in later studies. Maybe due to the plasma frequency ω_p^2 in BCS theory calculated by Anderson using random phase approximation [14] and later by Nambu through Ward identity [15] is coincident with BCS's calculation of "superfluid density" $\Pi(\mathbf{q}^2 \to 0) = \varepsilon_0\omega_p^2$, these two concepts are widely confused to be the same thing. In fact, BCS's calculation is not superfluid density but only plasma frequency. Historically, this misunderstanding had already existed within London equations $\partial_t \mathbf{J} = \Pi_1 \mathbf{E}$ and $\nabla \times \mathbf{J} = -\Pi_2 \mathbf{B}$: Londons assumed $\Pi_1 = \Pi_2$, resulting in the famous but obscure expression $\mathbf{J} = -\Pi\mathbf{A}$. [2]

The correct understanding of Meissner effect was first given by Ginzburg and Landau's (GL) theory, [20] i.e. symmetry breaking of a charged field coupled to electronmagnetic(EM) field. [21] Later, Anderson brought the idea to high energy physics to explain the mass of gauge particles, [19] known as the Anderson-Higgs mechanism. [22] In superconductors, the basic point is we have a coherent state $|\psi\rangle$ of a charged operator $\hat{\psi}$, i.e. $\hat{\psi}|\psi\rangle = \psi|\psi\rangle$ where the phase stiffness of ψ plays an essential role of Meissner effect. [23–25] Due to charge conservation, this kind of states can only exist in the sense of off diagonal long range order[26]. In the long wavelength limit, the GL Lagrangian reads

$$\mathcal{L}_{\mathrm{GL}} = \mathcal{L}_0 - \frac{1}{2M}|(\nabla + iqA)\psi|^2, \tag{2}$$

where M is a phenomenological parameter and q is the charge carried by ψ-field. From Eq. 2, we obtain the current $J = \frac{q}{2iM}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{M}\psi^*\psi A$. Take curl on two sides, we get the second London equation $\nabla\times\mathbf{J} = -\rho_s\mathbf{B}$, where the superfluid density $\rho_s = \frac{q^2}{M}\psi^*\psi$. This is the textbook's derivation of superfluid density from GL theory. On the other hand, due to Gorkov's pioneer work [27], people believe the GL theory is the correct low energy effective description of BCS theory, in consistent with symmetry analysis in hydrodynamic limit[28]. But we have seen that M is a phenomenological (undetermined) parameter. In practice, however, it is usually chosen by identifying $\rho_s = \frac{4e^2}{M}|\psi|^2$ (q is substituted by $2e$) with the BCS's original result $\rho_s^{\mathrm{BCS}} = \frac{ne^2}{m}$ without any exact proof.

Now let's consider the BCS wave function $|\mathrm{BCS}\rangle = \prod_k(u_k + v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger)|0\rangle$ with $|0\rangle$ the vacuum state. We should notice that there is a cutoff ω_D for Cooper pairs, i.e. $u_k(v_k) = 0(1)$ for $\varepsilon_k < -\omega_D$ and $u_k(v_k) = 1(0)$ for $\varepsilon_k > \omega_D$. Clearly, it is a coherent state satisfying $V\sum_k' c_{-k\downarrow}c_{k\uparrow}|\mathrm{BCS}\rangle = \Delta|\mathrm{BCS}\rangle$ ($\sum'$ means summation within energy shell $|\varepsilon_k| < \omega_D$) with V the pairing interaction and Δ the pairing order parameter. First, as a heuristic proof, we apply Landau's boost argument[27]. Suppose the condensation moves at velocity v with $\mathbf{A} = 0$, the total kinetic energy is $\frac{1}{2}N_s mv^2$ with N_s the number of paired electrons. On the other hand, from Eq. 2 we obtain the kinetic energy $\frac{V}{2M}|\psi|^2(2mv)^2$. Let these two energies equal, we get $\frac{|\psi|^2}{2M} = \frac{n_s}{8m}$, where $n_s = \frac{N_s}{V}$. Therefore, we obtain the superfluid density $\rho_s = \frac{4e^2}{M}|\psi|^2 = \frac{n_s e^2}{m}$. Here, it should be noted that Landau's boost argument can only be applied to condensed Cooper pairs but not all electrons to obtain the phase stiffness of the superfluid. Keep in mind only the phase of condensation determines superfluid density while electrons above ω_D have no well-defined phases. On the other hand, if we insist to apply Landau's boost method to all electrons, we will get another physical quantity: plasma frequency of the perfect conductor, which tells all electrons have no dissipation as a result of the energy gap.

Besides the above heuristic argument, a more formal derivation can be performed based on coarse graining in a quasiclassical way. [29] We divide the system into many microscopic-large and macroscopic-small blocks. The block size can be chosen as the pairing size $\xi = \frac{\pi v_f}{2\Delta}$. In each block, we can define a *local* field operator $\psi_{i\sigma} = \xi^{-3/2}\sum_k' c_{k\sigma}$. Then in the approximated coarse graining model, only local pairing $\Delta_i\psi_{i\uparrow}^\dagger\psi_{i\downarrow}^\dagger + H.c.$ is included. Thanks to the property of local pairing, the Lagrangian in long wave length limit can be obtained exactly. Repeat the derivation of Aitchison et al., [30] the superfluid density is found to be $\rho_s = \langle\psi_i^\dagger\psi_i\rangle\frac{e^2}{m} = \frac{n_s e^2}{m}$. From both the Landau's boost argument and the coarse graining derivation, we have arrived at our central conclusion: *superfluid density is determined by paired electrons but not all electrons. In contrast, plasma frequency (photon mass) is determined by all electrons. Therefore, BCS theory does explain the Meissner effect, but not in the same way as they showed in their milestone paper.* The superfluid density is given by $\rho_s = \frac{n_s e^2}{m}$ where n_s only accounts for paired electrons. In a rough estimation, $n_s = \left(\frac{\omega_D}{E_f}\right)n$ at zero temperature.

Of course, the above derivation only works for clean systems. In dirty limit, the plasma frequency (recognized as superfluid density before) is given by $\varepsilon_0\omega_p^2 = \Pi(\mathbf{q}^2 \to 0) = \pi\sigma_n\Delta$ [7] where σ_n should be understood as the normal state dc conductivity slightly above T_c. As a result, according to our theory, there is an additional

TABLE I. Summary of several scalings in hole-doped cuprate superconductors. σ_n and τ are dc conductivity and scattering rate in normal state slightly above T_c.

	underdoped	overdoped
$\left(\frac{\omega_D}{E_f}\right)$	~ 1	$\sim T_c$
τ	$\sim \frac{\hbar}{T^*}$	$\min\left(\tau_{\rm imp}, \sim \frac{\hbar}{T_c}\right)$
ρ_s	$\sim T_c$ (for small T_c, Uemura's law[8])	$\sim T_c$ (clean, Bozovic's law[1]) $\sim T_c^2$ (dirty, Bozovic's law[1])
ω_p^2	$\sim x$ (see e.g. [18])	$\sim \sigma_n T_c$ (clean, Homes' law[6]) $\sim \sigma_n$ (dirty, Drude behavior)

factor $\left(\frac{\omega_D}{E_f}\right)$ in superfluid density, so we have

$$\rho_s = \pi\left(\frac{\omega_D}{E_f}\right)\sigma_n\Delta \propto \left(\frac{\omega_D}{E_f}\right)\sigma_n T_c. \quad (3)$$

Here, we have replaced Δ by T_c in order to apply to underdoped cuprates, see below discussions. Interestingly, we can also apply Eq. 3 to clean superconductors as long as $\sigma_n = \frac{ne^2\tau}{m}$ with $\tau = \frac{\hbar}{\pi\Delta}$ known as the Plankian dissipation in literatures.[31] This universal scattering rate comes from the pairing interaction (due to phonon or spin-fluctuations, etc.) which dominates electron scattering processes near T_c. But clearly, it should have a cutoff $\tau_{\rm imp}$ due to scattering with impurities. In together, we have

$$\tau \sim \min\left(\tau_{\rm imp}, \frac{\hbar}{\pi\Delta}\right), \quad (4)$$

Combining Eq. 4 and Eq. 3, as long as $\tau \gg \frac{\hbar}{T_c}$, the superfluid density is not affected by impurity scattering (called clean limit below), just as Anderson theorem says. [32] Therefore, Planckian dissipation is not others, but a manifestation of Anderson theorem. On the other hand, when $\tau \ll \frac{\hbar}{T_c}$ impurity scattering strongly affects superfluid density as a property of dirty superconductors.

Our discussion about superfluid density can be generalized beyond BCS's description. First, let us examine the strong coupling superconducting theory. [33, 34] Now the fermion pairing has no cutoff in momentum space but in frequency space. As a result, we don't need $\left(\frac{\omega_D}{E_f}\right)$ in Eq. 3 but Δ should be understood as equal-time pairing, i.e. $\Delta(t=0) = \frac{1}{E_f}\int \Delta(\omega)d\omega \sim \left(\frac{\omega_D}{E_f}\right)\Delta(\omega=0)$, leading to no change of Eq. 3. Second, let us consider non-phonon mediated superconductors. As shown in Kohn and Luttinger's seminal work[35], pure Coulomb interaction between electrons can also induce superconductivity. In specific cases, the Coulomb interaction play roles through spin-fluctuations[36], charge-fluctuations or some other quantum fluctuations[37]. Here, *the Debye frequency ω_D should be recognized as an energy scale below which electrons pair and also condense.* Take spin-fluctuation superconductors [38] as an example: *high energy electrons with $|\varepsilon_k| > \omega_D$ participate in spin-fluctuations and serve as the pairing glue for low energy electrons with $|\varepsilon_k| < \omega_D$.* Such a picture applies to overdoped cuprates very well, see below discussions. On the other hand, in the underdoped side, due to the existence of pseudogap, Δ in Eq. 3 should be understood as superconducting order parameter with long range phase stiffness, or more exactly proportional to T_c. [39] While $\left(\frac{\omega_D}{E_f}\right)$ should be understood as a ratio of pairing electrons over total mobile carriers, which is reasonably supposed to be 1 due to enough high energy insulating electrons (e.g. local moments or valence bonds) mediating the pairing force. [40]

There are several implications of our theory on experiments. First, optical experiments have observed a universal scaling behavior named as Homes' law.[6] It says the superfluid density proportional to $\sigma_n T_c$ for almost all superconductors. Within our understanding, optical experiments only measure plasma frequency ω_p^2 but not superfluid density ρ_s. Therefore, the Homes' law should be updated to $\omega_p^2 \propto \sigma_n T_c$. Combining Drude theory, $\sigma_n = \varepsilon_0\omega_p^2\tau$, Homes' law implies $\tau \sim \frac{\hbar}{T_c}$, which is just the Planckian dissipation in clean superconductors. [41] As a result, Homes' law is expected to fail in dirty superconductors. Our re-explanation of Homes' law (and also FGT sum rule) is evidenced by direct measurement of superfluid density using mutual inductance technique by Bozovic et al. very recently.[1] They measured ρ_s upon doping in overdoped cuprates and found ρ_s proportional $T_c(T_c^2)$ for high(low)-T_c samples. This behavior is in contrast with the original Homes' law: *if* it is right, then combining two experiments we have σ_n independent

TABLE II. A comparison between plasma frequency and superfluid density in superconductors.

	plasma frequency ω_p^2	superfluid density ρ_s
London equation	$\partial_t \mathbf{J} = \varepsilon_0 \omega_p^2 \mathbf{E}$	$\nabla \times \mathbf{J} = -\rho_s \mathbf{B}$
EM property	perfect conductor $\mathbf{E} = 0$	perfect diamagnet $\mathbf{B} = 0$
related quantities	photon mass $m = \frac{\omega_p^2}{c^2}$	penetration depth $\lambda = \frac{1}{c}\sqrt{\frac{\rho_s}{\varepsilon_0}}$
energy scale	E_f	ω_D
need symmetry breaking?	no	yes

on doping for high-T_c samples and $\sigma_n \propto T_c$ for $T_c \to 0$, which is clearly wrong since overdoped cuprates behave like metal but not insulator. Moreover, direct measurement of dc conductivity showed the above deduction is incorrect. [1] Our work explains Bozovic et al.'s experiments very well. In high T_c regime, $\sigma_n \sim \frac{\omega_p^2}{T_c}$ (Homes' law or Planckian dissipation for clean superconductors), according to Eq. 3 we have $\rho_s \propto \omega_p^2 T_c$ (ω_D proportional to T_c has been used by supposing pairing interaction almost unchanged). In low T_c regime, $\sigma_n = \varepsilon_0 \omega_p^2 \tau_{\rm imp}$ is independent of T_c ($\tau_{\rm imp}$ is supposed unchanged upon doping), and thus we obtain $\rho_s \propto T_c^2$. For underdoped cuprates, according to the above discussion, $\left(\frac{\omega_D}{E_f}\right) \sim 1$ and Δ in Eq. 3 should be understood as T_c. Furthermore, ω_p^2 is proportional to carrier density x and τ is expected to obey a generalized Planckian dissipation $\tau \sim \frac{\hbar}{T^*}$ (since τ is related to pairing interaction and thus T^*). Therefore, we expect a scaling $\rho_s \sim x\frac{T_c}{T^*}$. In practice, since superconductivity only occurs as finite doping $x_c \sim 0.05$, such a scaling becomes $\rho_s \sim T_c$ approximately for small T_c, which has been observed in experiments known as Uemura's law.[8] The above discussions about cuprates are summarized in Table I.

Our work has disentangled two fundamental concepts in superconductors: plasma frequency and superfluid density. Their major differences are listed in Table II, and can be tested in future experiments for both conventional and unconventional superconductors, by comparing the plasma frequency obtained by optical conductivity and superfluid density obtained by directly measuring penetration depth (e.g. μSR or lower critical field H_{c1}). Here, from the theoretical point of view, we generalize the GL theory to capture their differences,

$$\mathcal{L}'_{\rm GL} = \mathcal{L}_{\rm GL} - \frac{1}{2} A_i \Pi^0_{ij} A_j, \tag{5}$$

where the second term comes from high energy unpaired electrons while the first term is the usual GL Lagrangian obtained by integrating out paired electrons. The plasma frequency is given by summing two contributions $\varepsilon_0 \omega_p^2 = \Pi^0(\mathbf{q}^2 \to 0) + \rho_s$. Interestingly, our theory can be generalized to relativistic case, which gives an additional term $\frac{1}{2} A_\mu \Pi^{\mu\nu} A_\nu$ to the Abelian Higgs model and thus an additional contribution to the gauge particle mass. In another word, the mass of the gauge particle may not only come from Higgs field but also from high energy plasma resonance. If this assumption is right, this high energy plasma mode is mediated by some unknown high energy particles. In the case of Yang-Mills theory, without coupling to EM field, these particles cannot be observed by light directly, reminiscent of the mysterious dark matter in cosmology. Therefore, it is interesting to ask whether the observed Higgs particle explains all the mass of W and Z gauge bosons?

TABLE III. Classification of several matter states from generalized London equations.

	ω_p^2	τ^{-1}	ρ_s
insulator	$= 0$	$\neq 0$	$= 0$
metal	$\neq 0$	$\neq 0$	$= 0$
perfect conductor	$\neq 0$	$= 0$	$= 0$
superconductor	$\neq 0$	$= 0$	$\neq 0$

At last, let us remark that our findings can be described by a generalized London equations:

$$\partial_t \mathbf{J} + \frac{\mathbf{J}}{\tau} = \varepsilon_0 \omega_p^2 \mathbf{E}, \quad \nabla \times \mathbf{J} = -\rho_s \mathbf{B}. \tag{6}$$

When $\tau = \infty$, it can be proved that when $\partial_t \mathbf{B} \neq 0$ these two equations are equivalent and $\rho_s = \varepsilon_0 \omega_p^2$. But in the static case $\rho_s \neq \varepsilon_0 \omega_p^2$ is allowed, as pointed in this work. Interestingly, based on the generalized London equations we can classify some matter states as shown in Table. III.

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[1] I. Bozovic, X. He, J. Wu, and A. T. Bollinger, Nature **536**, 309 (2016).
[2] F. London and H. London, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **149**, 71 (1935).
[3] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[4] J. E. Hirsch, Physica Scripta **80**, 035702 (2009).
[5] J. R. Schrieffer, *Theory of superconductivity* (Pennsylvania Univ., 1963).
[6] C. C. Homes, S. V. Dordevic, M. Strongin, D. A. Bonn, R. Liang, W. N. Hardy, S. Komiya, Y. Ando, G. Yu, N. Kaneko, X. Zhao, M. Greven, D. N. Basov, and T. Timusk, Nature **430**, 539 (2004).
[7] V. G. Kogan, Phys. Rev. B **87**, 220507 (2013).
[8] Y. J. Uemura, G. M. Luke, B. J. Sternlieb, J. H. Brewer, J. F. Carolan, W. N. Hardy, R. Kadono, J. R. Kempton, R. F. Kiefl, S. R. Kreitzman, P. Mulhern, T. M. Riseman, D. L. Williams, B. X. Yang, S. Uchida, H. Takagi, J. Gopalakrishnan, A. W. Sleight, M. A. Subramanian, C. L. Chien, M. Z. Cieplak, G. Xiao, V. Y. Lee, B. W. Statt, C. E. Stronach, W. J. Kossler, and X. H. Yu, Phys. Rev. Lett. **62**, 2317 (1989).
[9] P. A. Lee, N. Nagaosa, and X.-G. Wen, Rev. Mod. Phys. **78**, 17 (2006).
[10] J. Zaanen, Nature **536**, 282 (2016).
[11] M. Schafroth, Helv. Phys. Acta **24**, 645 (1951).
[12] M. R. Schafroth, Phys. Rev. **100**, 463 (1955).
[13] P. W. Anderson, Phys. Rev. **110**, 827 (1958).
[14] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[15] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[16] R. A. Ferrell and R. E. Glover, Phys. Rev. **109**, 1398 (1958).
[17] M. Tinkham and R. A. Ferrell, Phys. Rev. Lett. **2**, 331 (1959).
[18] D. N. Basov and T. Timusk, Rev. Mod. Phys. **77**, 721 (2005).
[19] P. W. Anderson, Phys. Rev. **130**, 439 (1963).
[20] V. L. Ginzburg, Rev. Mod. Phys. **76**, 981 (2004).
[21] Historically, Schafroth had applied his current-current correlation method to BEC and *successfully* explained Meissner effect. [12] This is misleading since in his case, all bosons condense and thus the plasma frequency $\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}$ obtained by current-current correlation exactly equals the superfluid density $\rho_s = \varepsilon_0 \omega_p^2$.
[22] P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964).
[23] B. D. Josephson, Phys. Letters **1**, 251 (1962).
[24] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).
[25] P. W. Anderson, *Basic notions of condensed matter physics* (Benjamin/Cummings Publishing Company, Advanced Book Program, 1984).
[26] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[27] A. Abrikosov, I. Dzyaloshinskii, and L. Gorkov, *Methods of quantum field theory in statistical physics* (Dover, 1975).
[28] M. Greiter, F. Wilczek, and E. Witten, Mod. Phys. Lett. B **03**, 903 (1989).
[29] In fact, BCS wave function can be recognized as a long range order only in the sense of coarse graining in a quasiclassical way, as pointed out by BCS themselves[3]: "In our theory, the correlation length is most reasonably interpreted as the distance over which the momentum of virtual pairs is the same. We believe that in this sense, the correlation length is effectively infinite.".
[30] I. J. R. Aitchison, P. Ao, D. J. Thouless, and X.-M. Zhu, Phys. Rev. B **51**, 6531 (1995).
[31] J. Zaanen, Nature **430**, 512 (2004).
[32] M. Ma and P. A. Lee, Phys. Rev. B **32**, 5658 (1985).
[33] G. Eliashberg, Sov. Phys. JETP **11** (1960).
[34] D. J. Scalapino, J. R. Schrieffer, and J. W. Wilkins, Phys. Rev. **148**, 263 (1966).
[35] W. Kohn and J. M. Luttinger, Phys. Rev. Lett. **15**, 524 (1965).
[36] D. J. Scalapino, E. Loh Jr, and J. E. Hirsch, Phys. Rev. B **34**, 8190 (1986).
[37] P. W. Anderson, Science **235**, 1196 (1987).
[38] D. J. Scalapino, Rev. Mod. Phys. **84**, 1383 (2012).
[39] F. C. Zhang, C. Gros, T. M. Rice, and H. Shiba, Supercond. Sci. Technol. **1**, 36 (1988).
[40] Interestingly, $\left(\frac{\omega_D}{E_f}\right)$ provides a criteria to justify the famous debate[42]: whether pairing glue is needed for high temperature superconductors? In the underdoped regime, $\left(\frac{\omega_D}{E_f}\right) \sim 1$ indicating all mobile electrons pair without the need of pairing glue. On the other hand, in the overdoped regime, $\left(\frac{\omega_D}{E_f}\right) < 1$. The low energy electrons pair though a glue mediated by high energy mobile electrons.
[41] In literatures, Homes' law was mostly explained as a behavior of dirty superconductors [7] which, however, implies an unphysical result: more dirty superconductors (less τ) with higher T_c.
[42] P. W. Anderson, Science **316**, 1705 (2007).